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Using maptrees to characterize topological change

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Abstract. This paper further develops the theory of maptrees, introduced in [**?**]. There exist well-known methods, based upon combinatorial maps, for topologically complete representations of embeddings of connected graphs in closed surfaces. Maptrees extend these methods to provide topologically complete representations of embeddings of possibly disconnected graphs. The focus of this paper is the use of maptrees to admit fine-grained representations of topological change. The ability of maptrees to represent complex spatial processes is demonstrated through case studies involving conceptual neighborhoods and cellular processes.

Keywords: maptree, topology, topological change, spatial information theory, conceptual neighborhood, cellular process

1 Introduction

The maptree was introduced in [**?**] as a structure that could represent fine topological details of spatial scenes. In this paper we exploit the maptree structure to provide an analysis of some types of spatial change.

By a spatial scene we mean an embedding of a graph in a surface. The general theory of maptrees applies to any orientable, closed surface embedded in Euclidean 3-space. These surfaces, according to the 1863 Möbius classification theorem for orientable, closed surfaces, are homeomorphic to a sphere with g handles (g-holed torus), for $g \geq 0$. The non-negative integer g is referred to as the *genus* of the surface. From henceforth we assume all surfaces are orientable. For the purposes of this paper, the only closed surface we consider is the sphere, and we show that the theory can also be applied to the Euclidean plane.

As discussed in [**?**], the maptree structure allows a finer level of detail of topological structure and relationships to be captured than some other approaches (for example, [**?**,**?**]). Similarly, in terms of change, the earlier theory developed by the author and colleagues in [**?**,**?**,**?**] allows, for example, a formal distinction between a hole emerging from a point in the center of a region and the same region merging with itself to create a hole, but cannot distinguish between a merge of two regions at a point or at a linear boundary. In this paper, we use the finer structural detail provided by the maptree to formally characterize some of the intricacies that can occur when a spatial scene undergoes topological change.

It appears that we have not gained anything by transforming one spatial configuration (a graph embedding) into another (a maptree). However, a maptree is a labelled tree, and hence unlike a general embedding it is a computationally tractable structure.

Section **??** of the paper reviews the salient definitions and properties of maptrees, both in the planar and spherical cases. This section draws on the corresponding material in [**?**], and is included for completeness, and for readers who may not be familiar with these constructions. However, we do include a new method of constructing maptrees from stars. In section **??**, we discuss topological change, focussing on merge and split operations and demonstrating the higher level of detail that can be captured. Section **??** presents two case studies on transitions in a well-known conceptual neighborhood and on a fundamental cellular process as demonstrators of the representational power of our approach. We conclude with a discussion and consideration of future work needed.

2 Background

The maptree construction brings together two separate formal descriptions of topological configurations, namely combinatorial maps [**?**,**?**] and adjacency trees [**?**,**?**]. We begin by developing the background theory of combinatorial maps.

2.1 Combinatorial maps

A *graph* is defined in the usual way as a set of vertices and edges between vertices, except that we allow edges that connect vertices to themselves, and also multiple edges between two vertices. A graph is *connected* if any pair of its vertices may be linked by a chain of adjacent edges. Informally, an *embedding* of a graph in a surface is a drawing of the graph on the surface in such a way that its edges may intersect only at their endpoints. Graph embeddings in closed surfaces have the property that the complement in the surface of an embedding of a connected graph is a collection of regions or *faces*, and each of these faces is a 2-manifold. If, furthermore, each of the faces is homeomorphic to a disc, the embedding is called a *2-cell embedding*. When the graph is embedded in the Euclidean plane, then one of the faces will be of infinite extent, and called the *external face*.

Let $A = \{a, b, \ldots, k\}$ be a finite collection of elements. We call any bijective function $\phi : A \to A$ a *permutation* of A. Essentially, we can think of ϕ as rearranging the elements of A. Now, any permutation can always be written as a collection of cycles $(a_1 a_2 \ldots a_n)$, where $a_2 = \phi a_1$, $a_3 = \phi a_2$, and so on, and $a_1 = \phi a_n$. So, for example, suppose $A = \{a, b, c, d, e\}$, and $b = \phi a$, $c = \phi b$, $a = \phi c$, $e = \phi d$, and $d = \phi e$. Then ϕ may be written in cycle notation as $\phi = (abc)(de)$.

Suppose now that we have a collection of permutations of A, $\Phi = \{\phi_1, \ldots, \phi_m\}$. Then Φ is *transitive* if, given any elements $x, y \in A$, we can transform x to y by a sequence of permutations from Φ. That is,

$$x \xrightarrow{\phi_{i_1}} x_1 \xrightarrow{\phi_{i_1}} \ldots x_p \xrightarrow{\phi_{i_p}} y$$

With these preliminaries taken care of, we can now define a combinatorial map.

Definition 1 *A* combinatorial map $\boldsymbol{M}\langle S, \alpha, \tau\rangle$ *consists of:*

1. *A finite set S of elements, called* semi-edges*, where the number of semi-edges is even. We can write S as $S = \{a, \overline{a}, b, \overline{b}, \ldots, k, \overline{k}\}$*
2. *A permutation α of S*
3. *A permutation τ of S, which in cyclic form is $\tau = (a\overline{a})(b\overline{b})\ldots(k\overline{k})$*

subject to the constraint that the collection of permutations $\{\tau, \alpha\}$ is transitive.

An example of a combinatorial map is given by $\mathbf{M}_1\langle S, \alpha, \tau\rangle$, where $S = \{a, \overline{a}, b, \overline{b}, c, \overline{c}\}$, $\alpha = (\overline{a}c\overline{b}\overline{c})(a)(b)$, and $\tau = (a\overline{a})(b\overline{b})(c\overline{c})$. It is easy to check that the collection $\{\tau, \alpha\}$ is transitive.

Combinatorial maps provide formal representations of 2-cell graph embeddings, as is seen from the following theorem due to Edmonds [**?**] and Tutte [**?**].

Theorem 1 *(Edmonds, Tutte) Each combinatorial map provides a topologically unique (up to homeomorphism of the surficial embeddings) representation of a 2-cell graph embedding in a closed surface. Conversely, every 2-cell graph embedding in a closed surface can be uniquely (up to permutation group isomorphism) be represented by a combinatorial map.*

Given a combinatorial map $\mathbf{M}\langle S, \alpha, \tau\rangle$, the 2-cell embedding is constructed as follows. Each edge of the embedded graph is represented by a pair of semi-edges, called a *facing pair*, transposed by τ. Each cycle of α defines the ordering of semi-edges around each face of the embedding. Each face is defined as the region on the left while traversing the semi-edges of a cycle of α. We say that combinatorial map $\mathbf{M}\langle S, \alpha, \tau\rangle$ *represents* the 2-cell embedding so constructed. Because of the transitivity property, any graph represented by a combinatorial map must be connected. We may also note that the constituent cycles of α are sufficient to uniquely reconstruct the embedding. The constituent cycles of α are termed the *α-cycles* of $\mathbf{M}$. Figure **??** illustrates this construction using combinatorial map, $\mathbf{M}_1\langle S, \alpha, \tau\rangle$.

We now focus on the nature of the closed surface in which the configuration represented by a combinatorial map is embedded (unique by Theorem **??**). To determine the surface from the combinatorial map, we need to determine the number of vertices of the embedded graph. To do this, we calculate a further permutation β by the formula $\beta = \tau\alpha^{-1}$ where the product is a composition of functions, and α^{-1} denotes the inverse function of α. In our example $\beta = (a\overline{a}c)(b\overline{b}\overline{c})$. We may note that each cycle of β represents a vertex of the embedded graph, and the cycle itself represents the ordering of semi-edges around the vertex.

Now we invoke the famous result of Euler and Poincaré:

Theorem 2 *(Euler-Poincaré) Given a 2-cell embedding of a graph in a surface of genus g, where V, E, and F are the numbers of vertices, edges, and faces, respectively. We have $V - E + F = 2 - 2g$.*

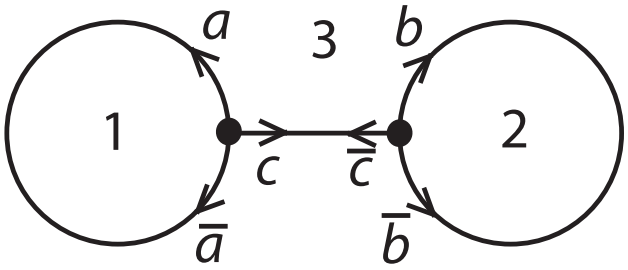

Fig. 1. Embedding of $\mathbf{M}_1$ in the sphere

Given a combinatorial map, $\mathbf{M}\langle S, \alpha, \tau\rangle$, the genus of its embedding surface can then be calculated by observing that V, E and F are the number of constituent cycles of β, τ and α, respectively. We sometimes refer to the genus of the combinatorial map, meaning the genus of its embedding surface. For the above map $\mathbf{M}_1\langle S, \alpha, \tau\rangle$, $V = 2, E = 3, F = 3$, and so $2 - 2g = 2$, and $g = 0$ which accords with our knowledge that the embedding surface is a sphere.

2.2 Star representation of a combinatorial map

As already observed, a combinatorial map is completely specified by its collection of α-cycles. This leads to the following visualization tool that will be helpful later.

Definition 2 *Given a combinatorial map* $\boldsymbol{M}$ *with* α*-cycles* $\alpha_1, \ldots, \alpha_n$*, then the* star *associated with* $\boldsymbol{M}$ *is an edge-labeled tree with a central black node from which edges connect to* n *white nodes, the* i*th edge being labeled with* α_i*,* $(1 \leq i \leq n)$*.*

White nodes represent faces of the embedding and the black node represents the connected network of edges. The star of example $\mathbf{M}_1$ is shown in figure **??**.

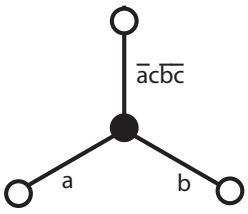

Fig. 2. Star representation of $\mathbf{M}_1$

2.3 Planar embeddings

The above results on combinatorial maps apply to closed surfaces, and therefore do not apply directly to the Euclidean plane. However, any graph that is embeddable in the sphere is embeddable in the plane, and conversely. The plane places an extra piece of structure on the embedding in that we have the notion of the infinite face. An embedding of a graph in the sphere may correspond to many topologically distinct embeddings of it in the plane, depending which spherical face becomes the infinite face. Figure **??** shows two non-homeomorphic planar embeddings representing $\mathbf{M}_1$. Of course, when viewed as embeddings on the sphere, they are homeomorphic. Note also, for reason of clarity, we show only one of each pair of semi-edges.

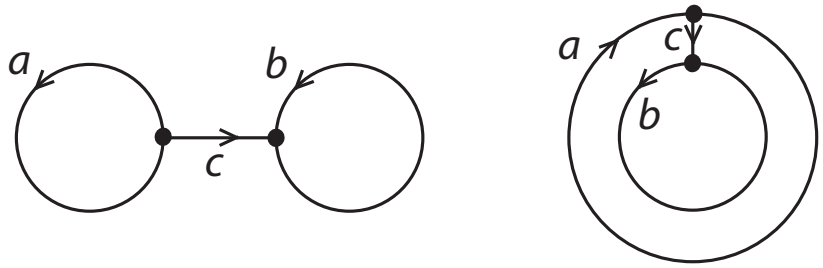

Fig. 3. Two homeomorphic embeddings of $\mathbf{M}_1$ in the sphere that are non-homeomorphic when viewed as planar embeddings

In order that a combinatorial map can uniquely specify a planar 2-cell embedding, all we need to do is specify which cycle represents the boundary of the external face. We can then invoke a slightly extended version of Theorem **??** (Edmunds, Tutte) to guarantee topological uniqueness of the representation.

With respect to the star representation, our approach is to make the white node representing the external face the root of a rooted tree. More formally, we have the following definition.

Definition 3 *Given a combinatorial map* $\boldsymbol{M}$ *with* α*-cycles* $\alpha_1, \ldots, \alpha_n$*, then a* p-star *associated with* $\boldsymbol{M}$ *is an edge-labeled rooted tree with a central black node from which edges connect to* n *white nodes, the* i*th edge being labeled with* α_i*,* $(1 \leq i \leq n)$*. The root of the tree is the white node whose incident edge is labeled by the infinite face.*

Viewed as a p-star, figure **??**, with the root node being by convention at the top, represents the planar embedding on the lefthand side of figure **??**.

2.4 Maptrees

Combinatorial maps have the limitation that they can only represent embeddings of connected graphs. (in fact, as we have said, every 2-cell embedding must be

the embedding of a connected graph). We now extend the algebraic representation of graph embeddings as combinatorial maps to also represent embeddings of disconnected graphs. Assume we have a collection of maximal connected components (henceforth called components) of a graph embedding, such that each of the components is a two-cell embedding. To specify the topology of the embedding, it is not sufficient to give the collection of stars (or p-stars) representing the components. To see this, consider the two graphs, each embedded in the surface of a sphere, shown on the lefthand and righthand sides of figure **??**. Both embeddings would have the same representation as a collection of three stars. However, it is easy to see that these embeddings are not topologically equivalent, in the sense that there does not exist a homeomorphism of the sphere that maps one embedding to the other. How graph embeddings stand with respect to one another becomes an issue, and we now develop the extra structure to represent this. The essential idea is to join the component stars together, merging them at appropriate white nodes.

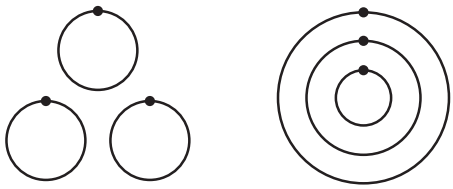

Fig. 4. Two non-homeomorphic embeddings in the sphere

The process is described in detail in [**?**], but here, we adopt a slightly different approach to the construction, illustrating with the embedding shown in figure **??**. In this figure we have labeled the faces $1, 2, \ldots$ (some no longer 2-cell) created by the embedding. We begin by considering stars of the components, as shown in figure **??**, where are also labelled the white nodes with the names of the regions that they represent. We then join the stars together, merging white nodes when they have the same face name. The resulting maptree is shown in figure **??**.

We are now ready to give the formal definition of a maptree. As a preliminary, we introduce the construction of a bw-tree, that is able to uniquely represent how the components and regions stand in the relation to each other, but does not provide details about the topology of the components themselves. This is a construction closely related to that of the adjacency tree, discussed in the next section.

Definition 4 *A* bw-tree ***X*** *is a colored tree with the nodes colored black or white, respectively, subject to the condition that no two adjacent nodes have the same color.*

We now combine the bw-tree construct with the combinatorial maps of the components.

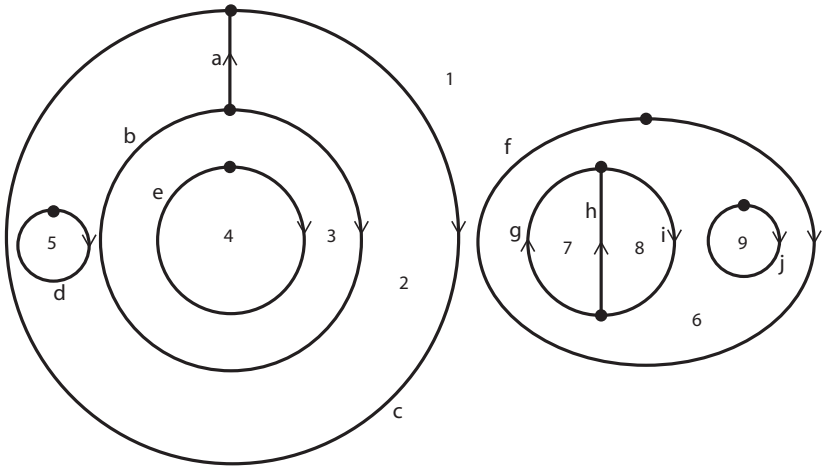

Fig. 5. Embedding of a disconnected graph

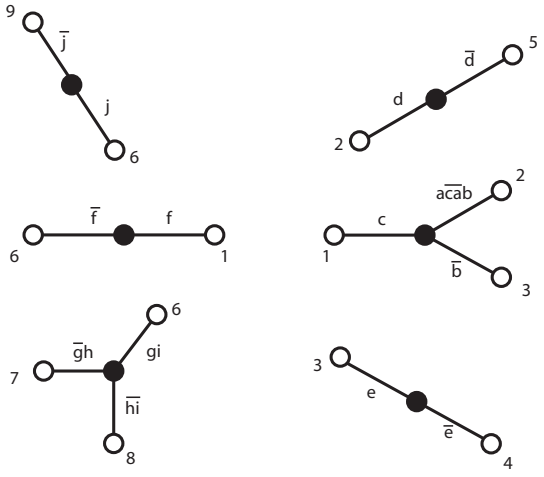

Fig. 6. Stars of the components of the embedding in figure **??**

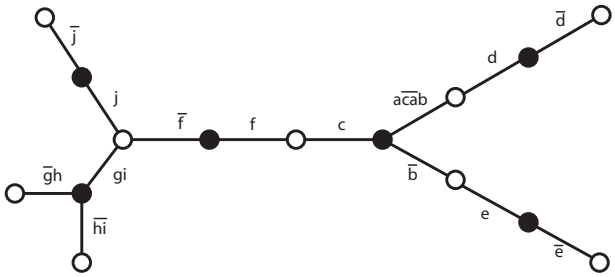

Fig. 7. Maptree of the embedding in figure **??**

Definition 5 *Let $\mathcal{M}$ be a finite collection of combinatorial maps, represented by stars. A* maptree *is an edge-labelled bw-tree $\mathcal{T}_{\mathcal{M}}$ formed by the merge of the component stars at white nodes.*

As above, the interpretation of maptree $\mathcal{T}_{\mathcal{M}}$ is that each black node of $\mathcal{T}_{\mathcal{M}}$ represents the 2-cell embedding of the map associated with that node. If two black nodes are connected via a white node, then the two cycles labeling the two edges joining the nodes represent the regions that "face up to each other" in the embedding.

We have the following proposition, discussed in more detail in [**?**].

Proposition 1 *Let $\mathcal{M}$ be a finite collection of combinatorial maps of genus zero, and $\mathcal{T}_{\mathcal{M}}$ be a maptree. Then $\mathcal{T}_{\mathcal{M}}$ provides a unique representation (up to homeomorphism of the sphere) of the disconnected graph embedding of the connected components represented by $\mathcal{M}$.*

2.5 Maptrees for planar embeddings

We can modify the map tree construction to account for planar embeddings by distinguishing the infinite exterior region, represented by one of the white nodes of one of the p-stars. We make this node the root of a rooted tree. The other p-stars are arranged under the root in such a way as their local roots connect with leaves of upper p-stars. As an example, the planar maptree for the planar embedding shown in figure **??** is shown in figure **??**, where the root is at the top of the figure.

3 Adjacency trees, maptrees, and topological change

In this section we relate topological change to structural changes to the maptree. We begin by throwing away some of the detail that maptree representations provide, and focusing on the nesting of regions inside each other. Insights gained are then applied to the more refined case. Adjacency trees were introduced by

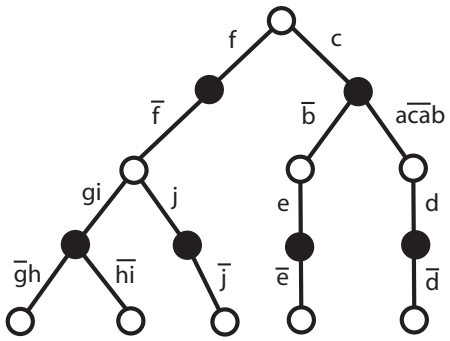

Fig. 8. Maptree of the planar embedding in figure **??**

Buneman [**?**] to represent the nesting structures of disk-like structures in the plane. As with all the work in this paper, their natural embeddings are in closed surfaces, but can be applied to the case of the plane by distinguishing the infinite face.

Definition 6 *Given an embedding of a collection of simple closed curves in a closed surface, its associated* adjacency tree *is a tree whose nodes are the faces of the embedding and where two nodes are joined by an edge if and only if the corresponding faces share a common bounding curve.*

Clearly, adjacency trees provide some useful characterization of surficial embeddings, but have no details about boundary intricacies. However they can be used to represent basic types of topological change (see [**?**,**?**,**?**]). For example, the upper part of figure **??** shows a region merging with itself and in the process engulfing some of its exterior space. (This is a planar embedding interpretation). The lower part shows the corresponding changes in the form of an unfolding operation on the adjacency tree. We may note that all these changes are reversible, as indicated by the double-headed arrows.

Figure **??** shows the merging of two regions and the corresponding folding operation on the adjacency tree. In this case the change is irreversible if we wish to recapture the qualities of the original regions from the merged region.

The third example, shown in figure **??** has similar start and end regions as the first, but this time the process is not self-merge but region insertion in the form of a hole in an existing region. The corresponding adjacency tree operation is the insertion of a new node and edge. This change is reversible, and results in a region deletion.

The first and third of these examples show that adjacency trees can capture the distinction between a hole forming from a self-merge and the insertion of a region.

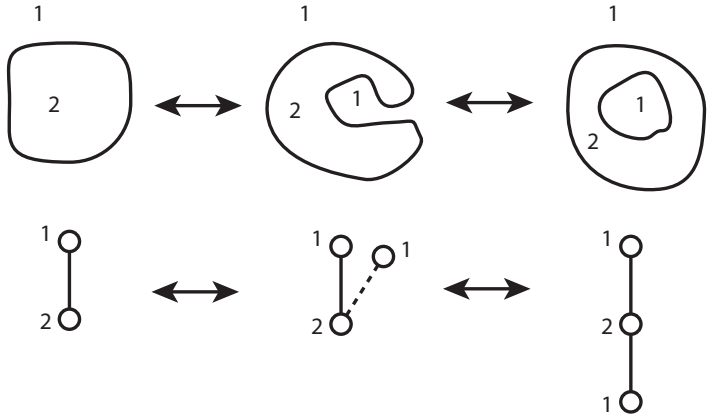

Fig. 9. Adjacency tree unfolding corresponding to a region self-merge

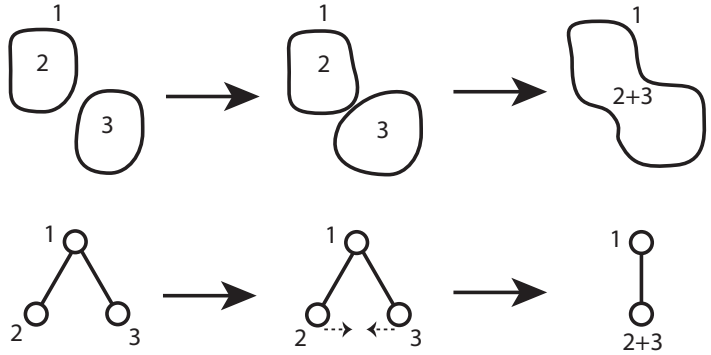

Fig. 10. Adjacency tree folding corresponding to the merge of two regions

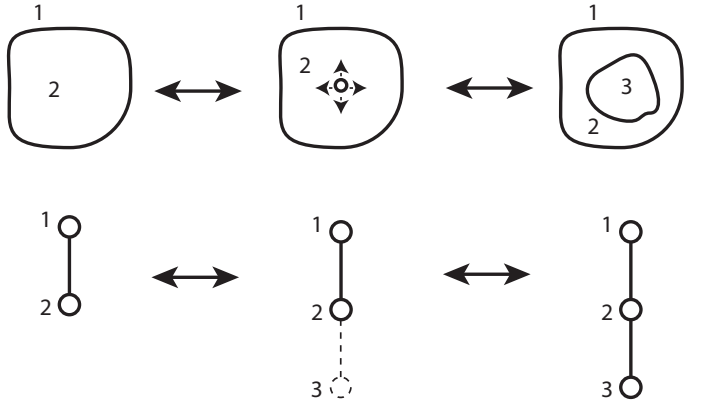

Fig. 11. Adjacency tree insertion corresponding to creation of a hole

3.1 Maptrees and topological change

We have indicated by means of examples how using folding/unfolding and insertion/deletion operations on adjacency trees can be used to represent basic kinds of changes to regions. We go on to demonstrate how maptree operations serve to represent in greater detail varieties of topological change. We focus on the operations of region merge and self merge, but the other cases are similar. These cases, which cover a large part of the atomic topological changes available to graph embeddings, give a flavor of the more find-grained kinds of analyses that maptrees facilitate. Throughout, we will use rooted maptrees, and hence be considering action in the plane. A very similar analysis follows in the spherical case.

Figures **??** and **??** are refinements of figure **??** in that in the self-merge of the region with itself we can distinguish merge at a point from merge at a line.

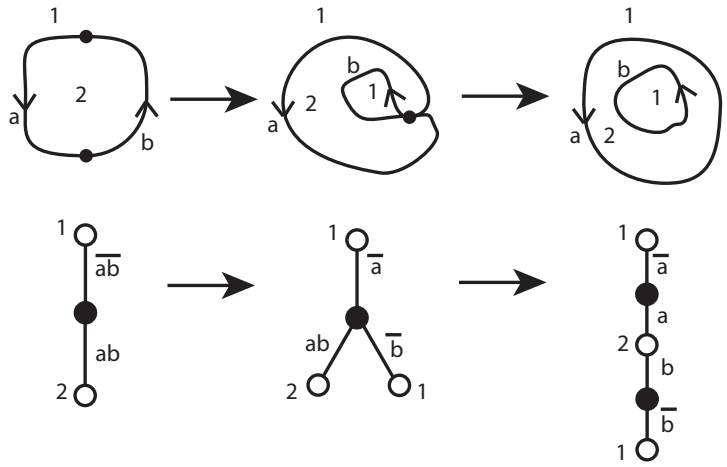

Fig. 12. Self-merge of a region at a point

In the first stage of the self-merge in figure **??**, shown in the middle section of the figure, the self-merge still leaves a single connected component, and the maptree star unfolds an edge and node from the top, using the central black node as fulcrum. The edge label $\overline{ab}$ splits into two labels $\overline{a}$ and $\overline{b}$. In the second stage of the self-merge, resulting in the formation of a hole, the single component splits into two. This is represented in the transformation of the maptree with an unfolding about lower black node labeled 2, where the edge label ab splits into two labels a and b .

The first stage of the line self-merge in figure **??** is a non-topological change bringing the edges c and d that will merge into closer proximity. In the next stage these edges merge, resulting in the single edge e. Formally, $d \mapsto e, c \mapsto \overline{e}$. The unfolding is similar to the previous example, except for the elimination of

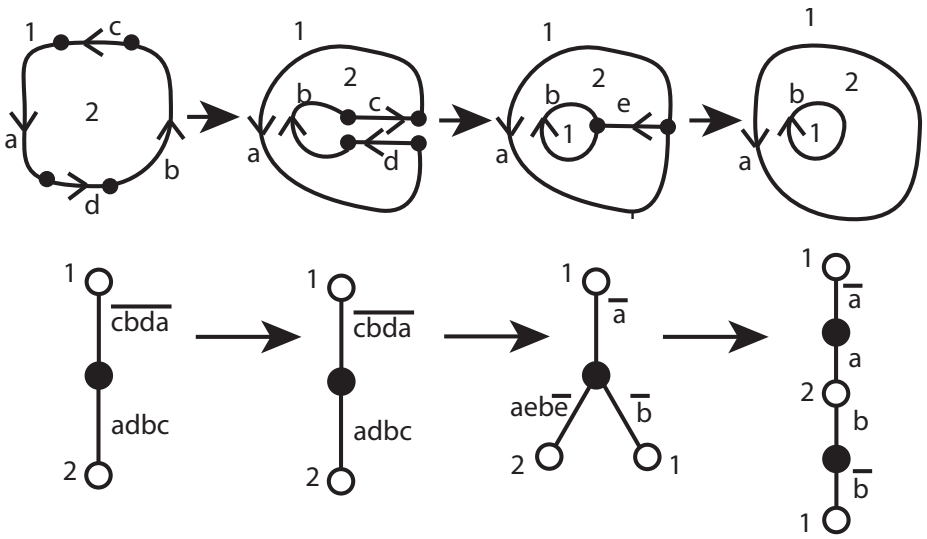

Fig. 13. Self-merge of a region at a line

$\overline{c}$ and $\overline{d}$ in the unfolding, and of e and $\overline{e}$ in the second. Again, these operations are reversible, with a self-split being represented by two foldings.

It is worth pointing out that if we work from right to left with either of the operations given in figures **??** and **??** but with focus on the two regions labeled 1, then the process of merge of the two regions, either at a point or a line, is represented by the folding operations given. Thus, region merges at a point or a line are represented similarly by folding/unfolding and edge label operations on maptrees.

4 Two case studies

In this section we develop two fairly substantial case studies that illustrate the utility of maptrees for representing topological change. The first might be called classical in that it provides a fine-grained representation of transitions in a well-known conceptual neighborhood diagram. The second provides an example of an fundamental process in cellular biology.

4.1 Conceptual neighborhoods revisited

Conceptual neighborhoods have been an important tool for spatial reasoning since their development by Freksa in [**?**] and use by many others (e.g., [**?**]). In this subsection we discuss the conceptual neighborhood diagram for the case of two regions We will see that maptrees provide a more fine-grained analysis of these planar topological changes than were obtained by earlier approaches. Figure **??** shows the diagram in question, and it provides both the region-region

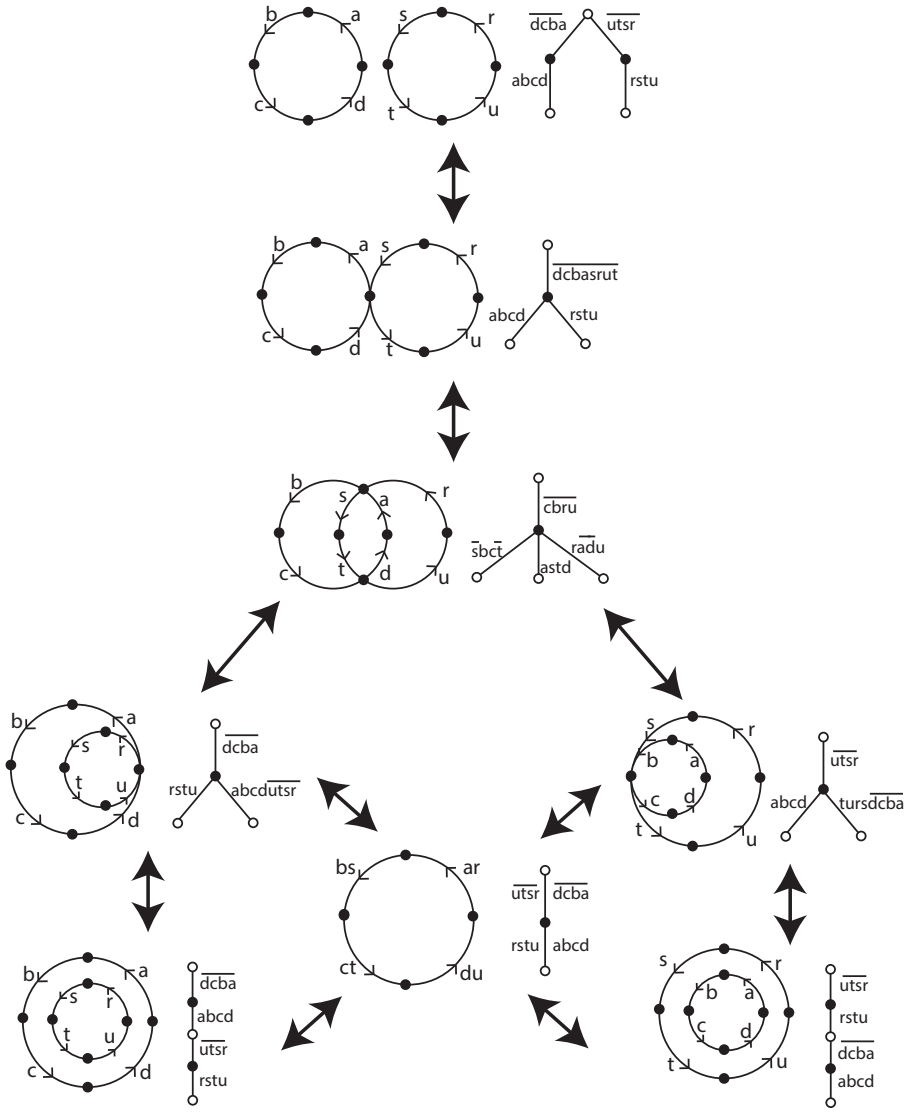

Fig. 14. Conceptual neighborhood with maptree representation

relationships and their maptree representation as disjoint regions meet at a point, overlap, are in a tangential proper part relationship at a point, are in a non-tangential proper part relationship, and are equal. Although we have not the space to show it, the maptree configurations would be different if the regions interacted at more than one point, at a line, or in a more complex manner. Thus the maptree representation can indeed capture more detail than RCC and other formulations.

As in the previous example, the transformations are represented in the maptree by various foldings and unfoldings along with edge label symbolic manipulations. The transition from disjoint to tangent regions is covered in the previous subsection, and is represented in the maptree by a folding around a black node and appropriate concatenation of cycles. The transition from tangent to overlapping regions is represented in the maptree by an unfolding of a new white node and some redistribution of edge labels. (This transition is actually a composition of rather simpler transpositions, but space precludes a full analysis here.) Transition to tangential proper part has some qualities of the inverse of the previous move, and the move to non-tangential proper part is similar to the inverse of the first transition from disjoint to tangent. We have shown the representation of equality of regions as having alternative labelings, based either on the edge labels a, b, c, d or r, s, t, u. We should note that the similarities to maptree operations in the previous section are no accident, as on the sphere, the transition from disjoint to tangent is identical to the transition from non-tangential to tangential proper part, as is the transition from overlapping to tangential proper part. In the case of the sphere, we lose the notion of a distinguished root node in the maptree.

4.2 Cellular topology

Our second illustration of the power of the maptree structure is in the modeling of part of the cellular process, phagocytosis, where solid particles are engulfed by the cell membrane to form an internal phagosome. Cellular processes have been the subject of much formal analysis (see, for example, [**?**,**?**]), however topological structure has not always been well-represented. A simplified form of the cellular transitions is shown in figure **??**. As we follow the process, we see foreign matter, bounded by edge (a) is disjoint and then tangent with the cell bounded by (bcd). The cell then makes geometrical changes in preparation to engulfing the foreign matter (no topological change) and then engulfs the foreign matter by firstly self-merging at a point and then thickening out its membrane. The foreign matter then detaches itself from the cell membrane and is enclosed in a vacuole (cd), called a phagosome, within the cell.

This process can be represented through all its stages by maptrees and their structural transformations, as shown in figure **??**. We have seen most of these processes already. The tree unfolds about its root, representing the tangential attachment of the foreign body to the cell. We have omitted the geometrical change to the cell as this results in no change to the tree. The self merge is represented by unfolding about the central black node, and thickening of the

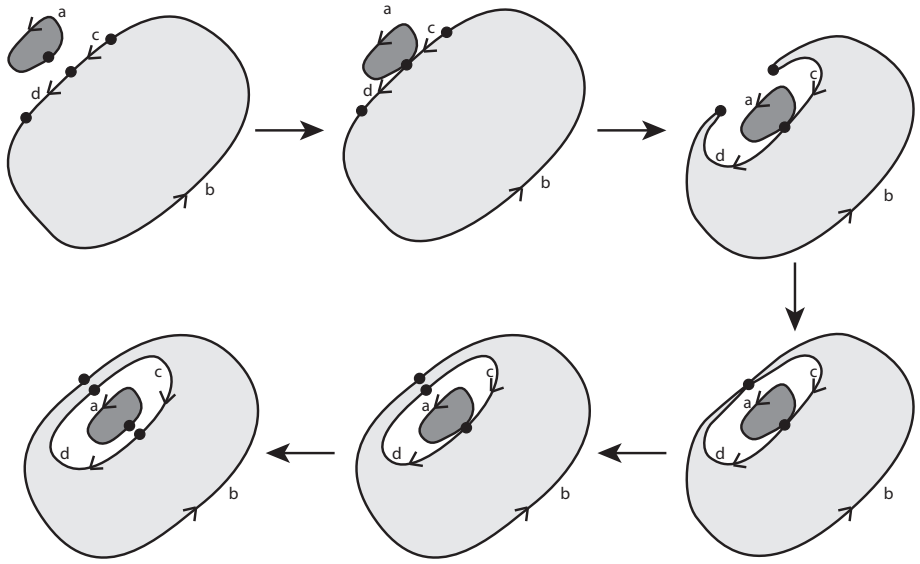

Fig. 15. Schematic of phagocytosis

membrane by unfolding about a white node. The final stage of detachment is represented by a further unfolding and results in a linear maptree structure.

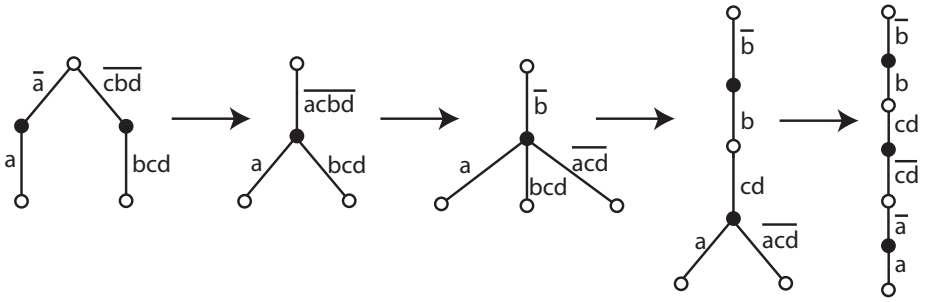

Fig. 16. Maptree transitions representing aspects of phagocytosis

5 Conclusions

This paper has further developed a topological representation of closed surficial embeddings that we call a maptree. We have demonstrated that the representation does indeed contain sufficient information to generate unique embeddings,

up to homeomorphism of the embedding surface. We have also shown how a simple modification provides a representation for planar embeddings. This work continues the line of work on formal description of spatial relationships found for example in [**?**,**?**]). However the map tree allows representation of multiple spatial regions, and allows a much greater level of complexity in the topological configurations to be represented.

We have also continued to develop a theory of topological change. We have illustrated how simple transitions between maptrees, involving foldings and unfoldings as well as an algebra of edge labels, can allow useful insights into the structure of basic topological changes. We have also illustrated its applications with two larger examples. The full theory, with its interplay between structural changes to the tree and its edge labels has yet to be fully elaborated, and this is the subject of future work. We also see great potential for maptrees in the formal descriptions of cellular processes, and this is the subject of current research.

Much more needs to be done in mining this rich vein of formal representation. Structures that can represent three dimensional configurations exist in the connected case, but add considerable complexity. The extension to disconnected embeddings as 3-manifolds is also a subject of ongoing research.

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